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Theoretical pressure–strain term, experimental comparison, and resistance to large anisotropy

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Although models of the pressure–strain term explain many features of nearly uniform homogeneous shear flows, a discrepancy remains (Leslie 1980). It is suggested that the discrepancy is caused by use of an empirical expression for the fluctuation part of the pressure–strain term, the part usually denoted by $\phi_{ij,1}$. The discrepancy is eliminated by replacement of the empirical $\phi_{ij,1}$ with a recent theoretical expression. Relatedly, the Launder, Reece & Rodi (1975) model for the mean-field part $\phi_{ij,2}$ is shown to be a good approximation for both a strongly and weakly sheared flow. This model of $\phi_{ij,2}$ when combined with the theoretical $\phi_{ij,1}$ is found to provide an explanation for experiments of both Champagne, Harris & Corrsin (1970) and Harris, Graham & Corrsin (1977). Full correction requires that deviations from local isotropy be accounted for. Special emphasis is given to a theoretical demonstration that the pressure–strain term does not cause a retun to isotropy but, rather, it resists large anisotropy – a weaker effect.

1. Introduction

It was pointed out by Leslie (1980) that the experiments of Champagne, Harris & Corrsin (1970, hereinafter referred to as CHC) and Harris, Graham & Corrsin (1977, hereinafter referred to as HGC) are of central importance in turbulence modelling, since the (difficult) pressure–strain term can be inferred directly from each of them. Such an inference is possible because of their quasi-homogeneous turbulence fields produced by a nearly uniform mean shear – the former at low shear, the latter at high shear. Leslie then showed that, although a model of the pressure–strain explained many features of these experiments, a major discrepancy remained. This implies a defect in the model, since the discrepancy is beyond any possible error in the measurement. The purpose of our article is to suggest wherein lies the defect in the model, and how it may be corrected.

The model actually consists of two parts. That is, Rotta (1951) showed that the pressure–strain term ϕ_{ij} can be divided into a fluctuation-field part denoted by $\phi_{ij,1}$ and a mean-field part denoted by $\phi_{ij,2}$. Both parts must be modelled, and there is some controversy as to which part has been adequately modelled and which not. The model of $\phi_{ij,1}$ most often used is an empirical model suggested by Rotta (1951). As to $\phi_{ij,2}$, the model most often used is that derived by Launder, Reece & Rodi (1975) and, from different considerations, by Naot, Shavit & Wolfshtein (1973). (There are actually two forms of the latter model – a general form and a simplification we refer to as the abridged form. The latter was used by Rodi (1976), Meroney (1976) and Gibson & Launder (1978).) Leslie assumed that the mentioned discrepancy between model and experiment is caused by use of the Launder *et al.* model of $\phi_{ij,2}$. Our suggestion is that the discrepancy is caused by use of the empirical $\phi_{ij,1}$. We are led to this suggestion by a recent theoretical expression for $\phi_{ij,1}$ derived by one of us (Weinstock 1981, 1982, hereinafter referred to as papers I and II respectively). One of our goals is to determine if the discrepancy between model and experiment is indeed eliminated when the empirical $\phi_{ij,1}$ is replaced by the theoretical expression.

We consider three related matters. (1) We wish to determine if either of the two $\phi_{ij,2}$ models (the general and abridged forms) is a good approximation for both experiments (CHC and HGC). If one model is correct the other is not. (2) We wish to demonstrate the difference between the empirical and theoretical $\phi_{ij,1}$, and the practical consequences: in particular, to point out that, whereas the former predicts a return to isotropy, the latter predicts a resistance to large anisotropy – a weaker effect. A comparison is also made with eddy simulations of $\phi_{ij,1}$. (3) We wish to determine if the predicted components of stress agree with experiment when the theoretical $\phi_{ij,1}$ is used. The method of algebraic modelling (Rodi 1976) is employed to calculate the stresses.

The organization of this paper is as follows. The difference between theoretical and empirical $\phi_{ij,1}$ is discussed in §2.1, where it is shown that the pressure–strain term causes a resistance to large anisotropy rather than return to isotropy. In §2.2 values of $\phi_{ij,2}$ are calculated from the experiments of HGC and CHC and compared with the models. These values are calculated by subtracting the theoretical $\phi_{ij,1}$ from the experimental ϕ_{ij} (i.e. $\phi_{ij,2} = \phi_{ij}(\text{exp}) - \phi_{ij,1}(\text{theory})$). In §3 stress components predicted by the theoretical $\phi_{ij,1}$ in combination with the general Launder *et al.* model of $\phi_{ij,2}$ are compared with experimental measurements. A summary and conclusion are given in §4.

An oversight in paper II one of us wishes to correct here is to acknowledge that a theoretical formalism closely related to that paper was presented by Cambon, Jaendel & Mathieu (1981). They emphasized a numerical analysis of shear flows. A statistical formalism for the pressure–strain term was also presented by Yoshizawa (1982) as an expansion in anisotropy.

2. Pressure–strain rate

2.1. *Fluctuation part: resistance to large anisotropy*

The purpose of this subsection is to introduce the fluctuation and mean field parts of the pressure–strain term, and to compare two expressions for the fluctuation part – the empirical expression and the recent theoretical expression.

For a nearly homogeneous shear flow, the Reynolds-stress equation can be written as

$$\frac{\mathrm{D}\langle u_i u_j\rangle}{\mathrm{D}t} = \underset{\text{production}}{P_{ij}} + \underset{\substack{\text{pressure}\\\text{-strain}}}{\phi_{ij}} - \underset{\text{dissipation}}{\epsilon_{ij}}, \tag{1}$$

where u_i is the ith component of the fluctuation velocity, x_i is a Cartesian coordinate, U_k is the mean velocity along the k-direction,

$$P_{ij} \equiv -\langle u_i u_k\rangle \,\partial U_j/\partial x_k - \langle u_j u_k\rangle\, \partial U_i/\partial x_k$$

is the turbulence production, ϕ_{ij} is the pressure–strain rate, ϵ_{ij} is the viscous dissipation, and we consider a unidirectional mean flow $\boldsymbol{U} \equiv \{U_1(x_2), 0, 0\}$ so that $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + U_1\,\partial/\partial x_1$.

Rotta (1951) showed that the pressure–strain-rate term ϕ_{ij} can be divided into two parts:

$$\phi_{ij} \equiv \left\langle p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle = \phi_{ij,1}+\phi_{ij,2}, \tag{2}$$

where p is the pressure fluctuation (ϕ_{ij} is denoted by $2A_{ij}$ in papers I and II). The first part $\phi_{ij,1}$ is a triple velocity correlation often referred to as the fluctuation part, or 'return-to-isotropy' part, while $\phi_{ij,2}$ is proportional to the mean-velocity gradient and is referred to as the mean-field part. The goal of theory and model is to express ϕ_{ij} in terms of $\langle u_i u_j \rangle$ – a single-point closure.

Most workers have used Rotta's model for the fluctuation part:

$$\phi_{ij,1} = -c_1 \frac{\epsilon}{k}(\langle u_i u_j\rangle - \tfrac{2}{3}k\delta_{ij}) \quad \text{(empirical model)}, \tag{3}$$

where $k \equiv \frac{1}{2}\langle u_i u_i\rangle$ is the turbulence-energy density, ϵ is the energy-dissipation rate and c_1 is the Rotta constant. Lumley (1978) has shown that c_1 cannot be constant. More recently, Weinstock in papers I and II calculated c_1 from first principles and showed that c_1 is not only not constant, but differs for different components ij. The theoretical expression for $\phi_{ij,1}$ is

$$\phi_{ij,1} = -C_{ij}^{(1)} \frac{\epsilon}{k}(\langle u_i u_j\rangle - \tfrac{2}{3}k\delta_{ij}) \quad \text{(theory)}, \tag{4}$$

where $C_{ij}^{(1)}$ are dimensionless coefficients determined explicitly by the theory. Expressions for these theoretical coefficients are given in the Appendix, where they are seen to vary with $\langle u_1^2\rangle/\langle u_2^2\rangle$ and $\langle u_1^2\rangle/\langle u_3^2\rangle$. This variation corresponds to, and accounts for, the ratios of the many scales of turbulence-energy spectra (i.e. $C_{ij}^{(1)}$ was derived from a two-point turbulence theory, and a relationship exists between $\langle u_1^2\rangle/\langle u_2^2\rangle$ and spectral scale ratios when the Reynolds number is large). For present convenience, the values of $C_{ij}^{(1)}$ are graphed in figures 1 (*a–c*) as functions of $\langle u_1^2\rangle/\langle u_2^2\rangle$ and $\langle u_1^2\rangle/\langle u_3^2\rangle$ for the case of local isotropy. A deviation from local isotropy is accounted for in figure 2. These figures show that $C_{ii}^{(1)}$ varies greatly with anisotropy, and that $C_{11}^{(1)} \neq C_{22}^{(1)} \neq C_{33}^{(1)}$. Both features are in sharp contrast to the Rotta model (3). Both features were also found in eddy simulations by Feiereisen, Reynolds & Ferziger (1981). (That $C_{ij}^{(1)}$ can even be negative for some conditions is not without precedent, since negative values were implied in the unstable atmospheric boundary layer (Wyngaard 1980).) The difference between theoretical and empirical values of $C_{ij}^{(1)}$, and an implication for turbulence modelling, is shown in §3. However, the most striking feature of $C_{ij}^{(1)}$, and the feature we stress here, has to do with the concept of return to isotropy. It is seen in figure 1 (*a*) that $C_{11}^{(1)}$ is relatively quite small at small anisotropy and increases with anisotropy. For example, $C_{11}^{(1)}$ increases from 0.6 to 1.1 as $\langle u_1^2\rangle/\langle u_2^2\rangle$ increases from 1 to 4, with $\langle u_1^2\rangle/\langle u_3^2\rangle$ held fixed at unity. Since, as pointed out by Lumley (1978), the pressure–strain term does not cause decaying turbulence to be isotropic unless $C_{11}^{(1)} > 1$, we conclude from figure 1 (*a*) that there is no return to isotropy. Rather, there is only a resistance to large anisotropy.

If, indeed, the empirical modal (3) is inaccurate – as we believe – it might be the root cause of the discrepancy between model and experiment. Our object is to determine if this is so by replacing (3) with the theoretical expression, and comparing its prediction with experiment. The underlying assumption is that one of the Launder *et al.* models is a good approximation for $\phi_{ij,2}$. Let us next see if this assumption is true for HGC and CHC.

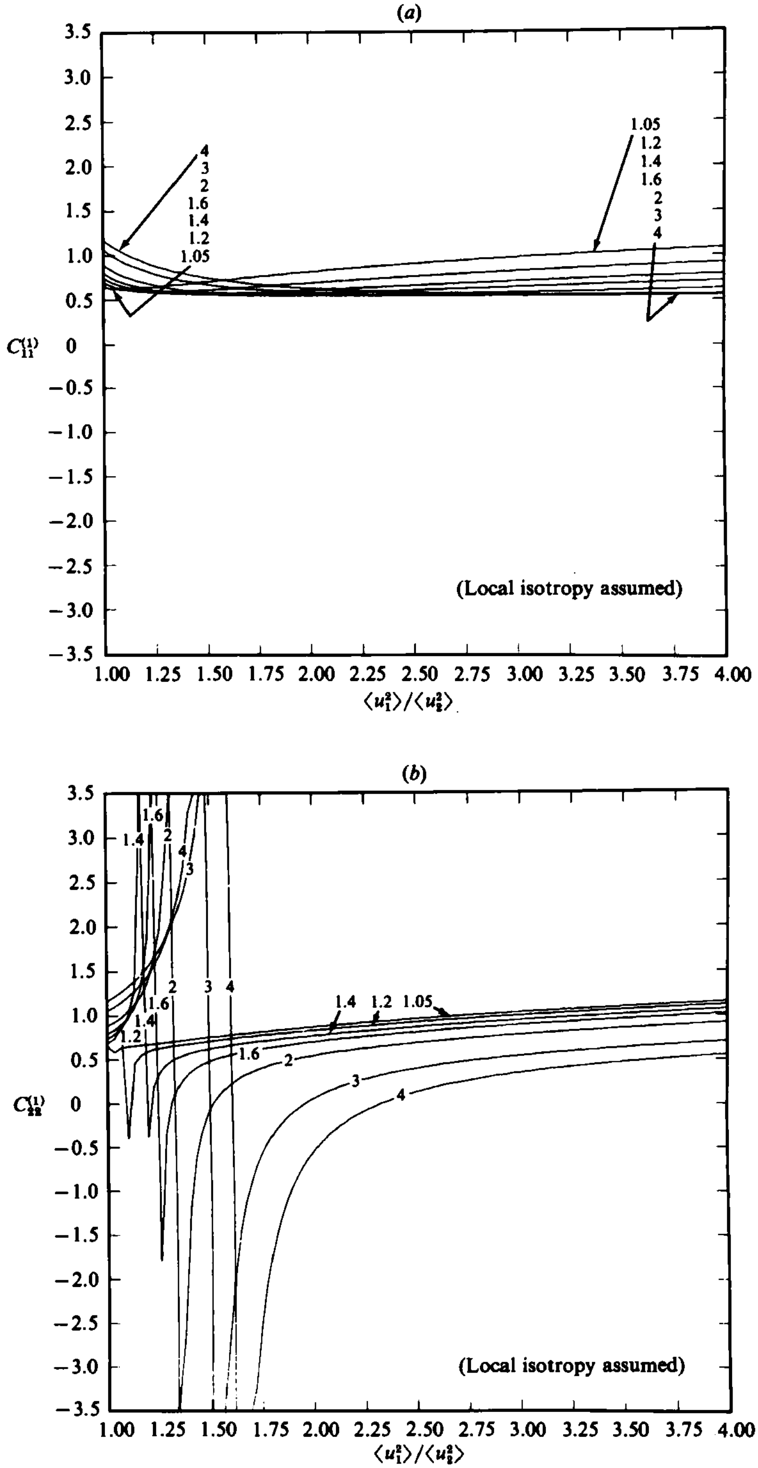

FIGURE 1 (a, b). For caption see facing page.

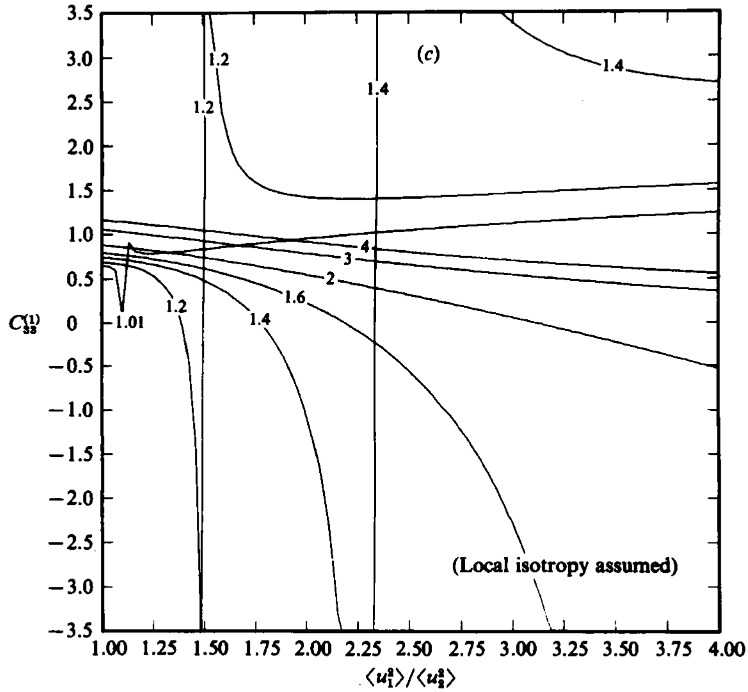

FIGURE 1. Theoretical $C_{ii}^{(1)}$ versus $\langle u_1^2\rangle/\langle u_2^2\rangle$ for various values of $\langle u_1^2\rangle/\langle u_3^2\rangle$: (*a*) $C_{11}^{(1)}$; (*b*) $C_{22}^{(1)}$; (*c*) $C_{33}^{(1)}$. These calculations are for the case of local isotropy, and consequently do not apply to CHC and HGC. Figure 2 should be used for those experiments. (Rapid variations of $C_{ii}^{(1)} = -(k/\epsilon)\,\phi_{ii,1}/a_{ii}$ seen in (*b*) and (*c*) occur in neighbourhoods of $a_{ii} = 0$ (vanishing anisotropy). The strain-rate terms $\phi_{ii,1}$ are small near those points, but do not necessarily vanish. The latter behaviour differs from the Rotta model, which has $\phi_{ii,1}$ vanish whenever a_{ii} vanishes.)

2.2. *Mean-field part*

The purpose of this subsection is to determine if either of the models of $\phi_{ij,2}$ is compatible with the data of both the HGC and CHC experiments, i.e. to verify if either model provides a good approximation for both strong and weak shears. A direct procedure for this verification would be to calculate $\phi_{ij,2}$ for both experiments and compare with the models. However, these experiments actually determine the total pressure–strain term $\phi_{ij} \equiv \phi_{ij,1}+\phi_{ij,2}$ and not $\phi_{ij,2}$ by itself. Hence to obtain experimental values of $\phi_{ij,2}$ we must subtract off the fluctuation part $\phi_{ij,1}$. Our contention is that the Rotta model of $\phi_{ij,1}$ is not satisfactory for this subtraction, and that the theoretical expression (4) should be used instead. On substitution of (4) into (2), the mean-field part is written as

$$\phi_{ij,2} = \phi_{ij}+C_{ij}^{(1)}\frac{\epsilon}{k}(\langle u_i u_j\rangle-\tfrac{2}{3}k\delta_{ij}). \qquad (5)$$

This equation gives the mean-field part in terms of measurable quantities and theoretical coefficients $C_{ij}^{(1)}$. It allows $\phi_{ij,2}$ to be determined from measurements of HGC and CHC. However, experimental values of $\phi_{ij,2}$ are of little interest in themselves. They are important if relatable to a general formulation of $\phi_{ij,2}$ – such as the Launder *et al.* models. Our goal, then, is to compare the experimental values of $\phi_{ij,2}$ in (5) with the models of $\phi_{ij,2}$.

To facilitate this comparison, we note that both forms of the models (the general and abridged forms) can be written – for our case of a simple shear – as

$$\phi_{ij,2} \equiv -C^{(2)}_{ij}(P_{ij}-\tfrac{2}{3}P\delta_{ij}), \qquad (6)$$
$$P \equiv \tfrac{1}{2}P_{11}.$$

Here $C^{(2)}_{ij}$ are dimensionless coefficients. The two models differ only in the values of these coefficients: for the abridged model $C^{(2)}_{ij} = c_2 =$ constant (all i,j), whereas for the general model, for a unidirectional flow, $C^{(2)}_{11} = \frac{1}{11}(9-3\gamma)$, $C^{(2)}_{22} = \frac{1}{11}(12-15\gamma)$, $C^{(2)}_{33} = \frac{1}{11}(6+9\gamma)$ (Launder *et al.* 1975). From a more general point of view, (6) can be viewed as a definition of $C^{(2)}_{ij}$. From that point of view $C^{(2)}_{ij}$ can vary with the mean shear, and need not be constant. The questions then are: (*a*) do either of the two models give a good approximation for $C^{(2)}_{ij}$; and (*b*) does $C^{(2)}_{ij}$ vary with the dimensionless shear? It is to help answer these questions that we calculate these coefficients for the experiments of both HGC and CHC and compare with the models. The expression employed for these calculations of $C^{(2)}_{ij}$ is given by combination of (5) and (6):

$$-C^{(2)}_{ij}(P_{ij}-\tfrac{2}{3}P\delta_{ij}) = \phi_{ij}+C^{(1)}_{ij}\frac{\epsilon}{k}(\langle u_i u_i\rangle-\tfrac{2}{3}k\delta_{ij}). \qquad (7)$$

We estimate the diagonal elements $C^{(2)}_{ii}$ first for the HGC experiment and afterwards for the CHC experiment. Expressions for the theoretical coefficients $C^{(1)}_{ii}$ are given in the Appendix, and, for convenience of computation, their values for the experiments are given in figures 2 (*a–c*). These expressions are taken from paper II.

To determine $C^{(1)}_{ii}$ we substitute the HGC experimental values $\langle u_i^2\rangle = (4100, 1630, 2450)$ cm^2 s^{-2} for $i = 1, 2, 3$ in figure 2 and find $C^{(1)}_{ii} = (0.98, 1.17, 0.15)$ for $i = 1, 2, 3$. (Figure 2 differs from figure 1 because local isotropy is not assumed – as explained in §3.) These theoretical values of $C^{(1)}_{ii}$ are notably consistent with eddy-simulation values (see table 4.5 of Feiereisen *et al.* 1981) – as are the CHC values given before (12). These values are markedly different from the coefficient $c_1 \approx 2$ widely used in models.

To determine the diagonal elements ϕ_{ii} for the HGC experiment, we write (1) as

$$\phi_{ii} = \frac{\mathrm{D}\langle u_i u_i\rangle}{\mathrm{D}t} - P_{ii}+\epsilon_{ii} \quad (i = 1, 2, \text{ or } 3). \qquad (8)$$

We rely on Leslie's computation of ϕ_{ii}, except for one difference, namely we do not assume the local-isotropic relation $\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}$. Thus, from (8),

$$\phi_{ii} = \phi^0_{ii}+(\epsilon_{ii}-\tfrac{2}{3}\epsilon), \qquad (9)$$

where ϕ^0_{ii} are the local-isotropic values computed by Leslie. An estimate of ϵ_{ii} is obtained from the spectral data in figure 21 of CHC, which show that $F_i(k_1) \propto \epsilon^{\frac{2}{3}}_{ii} k_1^{-\frac{5}{3}}$, with $\epsilon_{11} > \epsilon_{33} > \epsilon_{22}$. Such anisotropy of dissipation was also found in large-eddy simulations by Feiereisen *et al.* (1981). (Anisotropy of ϵ_{ij} is also found in measurements in the atmospheric surface layer under conditions of neutral stability (Kaimal *et al.* 1972), although there the ratio of ϵ_{22} to ϵ_{11} is influenced by the surface as well as the shear.) The CHC values (their figure 21) indicate $\epsilon_{11} = (1.2$ to $1.3)\,\epsilon_{22}$ and $\epsilon_{33} = (1.1$ to $1.15)\,\epsilon_{22}$ in the (nearly) inertial subrange. This magnitude of deviation from local isotropy (about 25%) conforms to that found by CHC and HGC from other considerations (see §5.2 of CHC or §4.7 of HGC). Approximate ratios of ϵ_{ii} to the measured ϵ are determined by $\epsilon_{11}+\epsilon_{22}+\epsilon_{33} = 2\epsilon$ (with $\epsilon_{11} \approx 1.25\epsilon_{22}$, $\epsilon_{33} \approx 1.13\epsilon_{22}$) to be

$$\frac{\epsilon_{11}}{\epsilon} = 0.73, \quad \frac{\epsilon_{22}}{\epsilon} = 0.59, \quad \frac{\epsilon_{33}}{\epsilon} = 0.68 \tag{10}$$

– quite comparable to the eddy-simulation values (Feiereisen *et al.* 1981). These corrections for ϵ_{ii} are combined with the computation of Leslie given by $\phi^0_{ii} = (-7.21, 3.39, 3.83) \times 10^4\ \mathrm{cm^2\,s^{-3}}$ (for $i = 1,2,3$), to obtain $\phi_{ii} = (-7.15, 3.31, 3.84) \times 10^4\ \mathrm{cm^2\,s^{-3}}$. The anistropy correction is seen to be negligible for the HGC experiment, but is found to be significant for the CHC experiment calculated in the next paragraph. (Also, in both experiments, the correction is quite significant for $C^{(1)}_{ii}$, as shown in the Appendix.) Finally, the values of $C^{(2)}_{ii}$ for HGC are given by substitution into (7) of this ϕ_{ii} together with the experimental values $\epsilon = 3.76 \times 10^4\ \mathrm{cm^2\,s^{-3}}$ (as corrected by Leslie 1980), $k = 4090\ \mathrm{cm^2\,s^{-2}}$, $P = 5.84 \times 10^4\ \mathrm{cm^2\,s^{-3}}$, and the values of $C^{(1)}_{ii}$ and $\langle u_1^2 \rangle$ given above. The result is

$$C^{(2)}_{11} = 0.73, \quad C^{(2)}_{22} = 0.50, \quad C^{(2)}_{33} = 0.96 \quad \text{(HGC)}. \tag{11}$$

Upon comparison, it is seen that these values are quite close to the general model (within 5 %) when γ is selected to be 0.42. With regard to the abridged value, there is a definite, significant difference – no matter what value is chosen for c_2. (This difference is alluded to by Leslie, who pointed out that the abridged model cannot explain the observed (2,3)-plane anistropy because it has $C^{(2)}_{22} = C^{(2)}_{33}$).) Next we determine the CHC values of $C^{(2)}_{ii}$.

To calculate $C^{(2)}_{ii}$ for the CHC experiment, we proceed in the same manner as done for HGC. The downstream values $\langle u_i^2 \rangle = (500, 260, 300)\ \mathrm{cm^2\,s^{-2}}$ are substituted in figure 2 to obtain $C^{(1)}_{ii} = (1.02, 1.32, 0.52)$. These values bear a remarkable resemblance to average eddy-simulation values (table 4.5 of Feireisen *et al.* 1981). Since advection is practically zero for CHC, the pressure–strain is simply $\phi_{ii} = -P_{ii}+\epsilon_{ii}$. It is easily evaluated by substitution of the experimental $\epsilon = P = 2.35 \times 10^3\ \mathrm{cm^2\,s^{-3}}$ together with the derived values of ϵ_{ii}/ϵ given by (10) to obtain $\phi_{ii} = (-3.07, 1.486, 1.556) \times 10^3\ \mathrm{cm^2\,s^{-2}}$. The desired coefficients $C^{(2)}_{ii}$ are then obtained by substitution of this ϕ_{ii} and $C^{(1)}_{ii}$ into (7) together with experimental values of ϵ and $\langle u_i^2 \rangle$ and $k = 530\ \mathrm{cm^2\,s^{-2}}$. The result is

$$C^{(2)}_{11} = 0.725, \quad C^{(2)}_{22} = 0.53, \quad C^{(2)}_{33} = 0.92 \quad \text{(CHC)}. \tag{12}$$

These values are seen to be remarkably close to the HGC values in (11) – within 5 %. The agreement between CHC and HGC values implies that $C^{(2)}_{ii}$ does not vary appreciably with the magnitude of mean shear or, more meaningfully, with the dimensionless shear strength $(k_0 v_0)^{-2}\ (\partial U_1/\partial x_2)^2$, where $v_0 \equiv (\frac{2}{3}k)^{\frac{1}{2}}$ and k_0 is a wavenumber characteristic of the energy-containing subrange, since the *magnitude* of shear has significance only in relation to turbulence velocity and scale lengths. The dimensionless shear varies by a substantial amount between the two experiments – a factor of 5. Perhaps of even greater interest is that the experimental values of $C^{(2)}_{ii}$ are all within 5 % of the general Launder *et al.* model when γ is selected to be 0.42. To us, these coincidences suggest that the aforementioned values of $C^{(2)}_{ii}$ are correct, to within 5 % for quasi-homogeneous shear flow, and are suitable for strong shears as well as weak. Variations of a model $\phi_{ij,2}$ were found in eddy simulations (Feiereisen *et al.* 1981), but for a different model than that of Launder *et al.* Although Leslie has proved that the general model is not exact, the error he finds is, on the whole, less than 10 % when $\gamma = 0.4$.

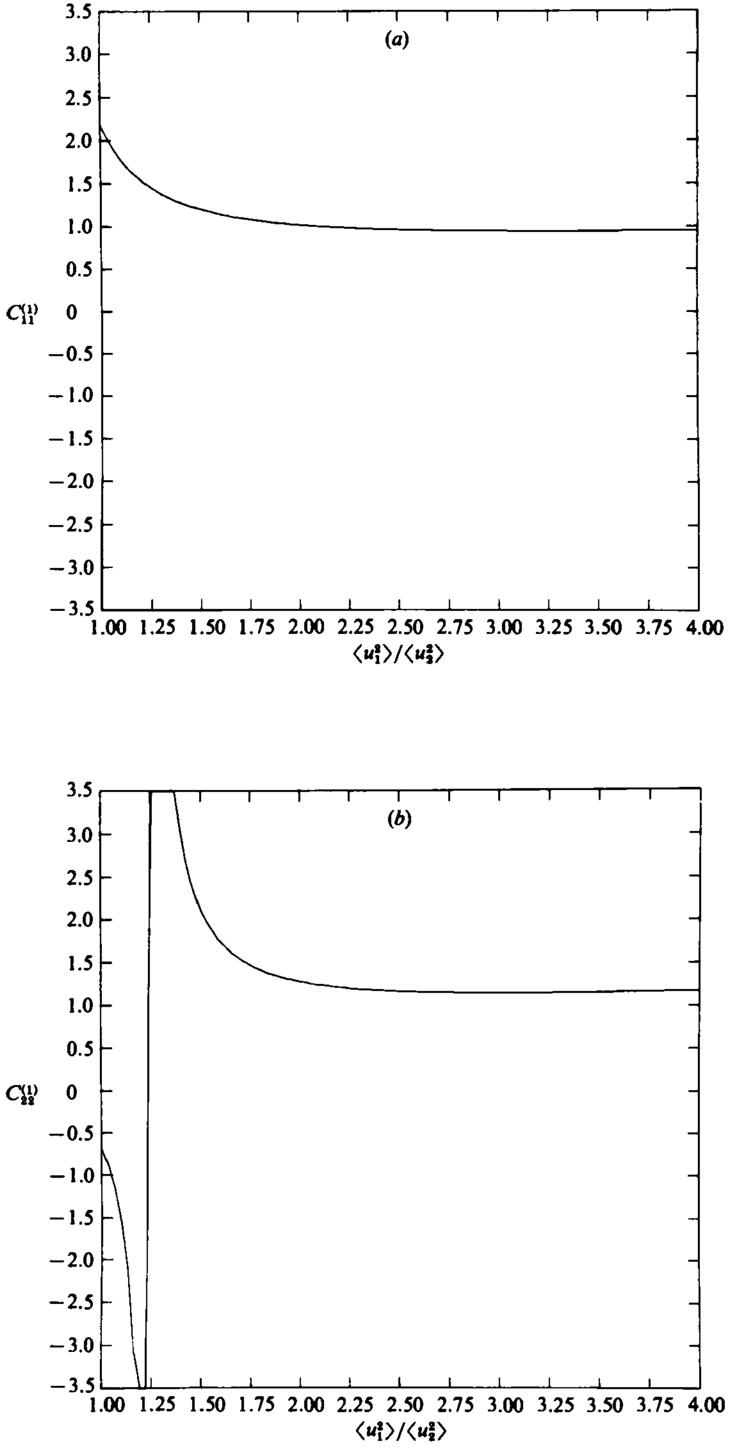

FIGURE 2 (a,b). For caption see facing page.

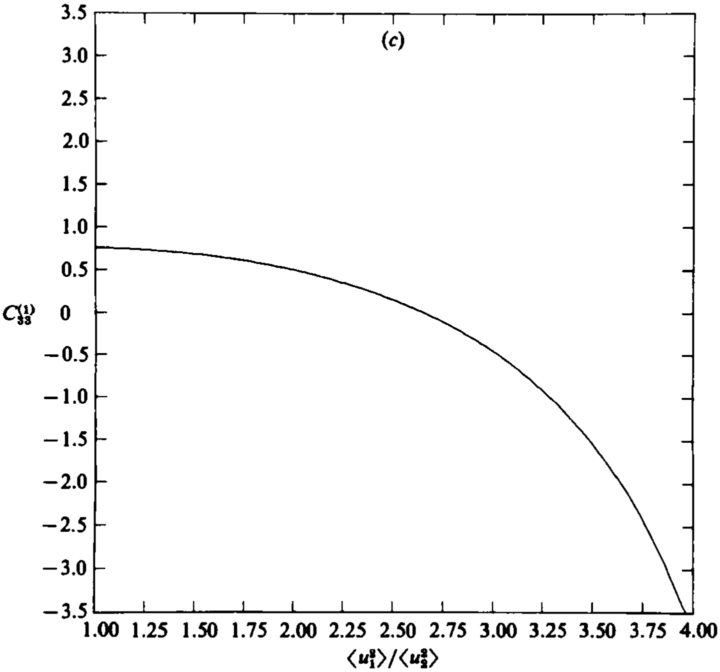

FIGURE 2. Theoretical $C^{(1)}_{ii}$ versus $\langle u_1^2\rangle/\langle u_2^2\rangle$ at $\langle u_1^2\rangle/\langle u_3^2\rangle = 1.67$ with deviations from local isotropy given by $\epsilon_{11} = 1.3\epsilon_{22}$ and $\epsilon_{33} = 1.13\epsilon_{22}$ – the value for CHC and presumably for HGC.

To complete our discussion of $C^{(2)}_{ii}$, there only remains to determine the off-diagonal element. This is easily done, since (7) gives $C^{(2)}_{12}$ as

$$C^{(2)}_{12}\langle u_2^2\rangle\frac{\partial U_1}{\partial x_2} = \phi_{12} + C^{(1)}_{12}\frac{\epsilon}{k}\langle u_1 u_2\rangle, \tag{13}$$

where we have used $P_{12} = \langle u_2^2\rangle \partial U_1/\partial x_2$. The experimental values of ϕ_{12} are $7.21\times10^4\ \mathrm{cm^2\,s^{-3}}$ for HGC (see Leslie) and $3.220\times10^3\ \mathrm{cm^2\,s^{-3}}$ for CHC (where $\phi_{12} = -P_{12}$ in the latter experiment, since there is no advection). The theoretical value of $C^{(1)}_{12}$ is 1.7 ± 0.2 (see paper I), and we will use 1.8. Substitution of these values in (13) together with the corresponding measured values of ϵ, k, $\langle u_2^2\rangle$ and $\langle u_1 u_2\rangle$, we obtain

$$C^{(2)}_{12} = \begin{cases} 0.55 & \text{(CHC)},\\ 0.64 & \text{(HGC)}.\end{cases} \tag{14}$$

These values agree approximately with the general model value ≈ 0.6. The abridged model is also in agreement. Note, however, that the variation of $C^{(2)}_{12}$ with mean shear is almost 15% – a distinctly larger variation than that of the diagonal elements with shear. This variation is still relatively small, since the dimensionless shear parameter varies by a factor of 5, and use of the median value 0.6 would lessen possible discrepancies to less than 10%. The value 0.6 is also obtained from purely theoretical considerations (Crow 1968) – albeit for very small anisotropy.

In summary, it is seen that the experiments tend to confirm the general Launder *et al.* model as a good approximation for high as well as low shear with γ selected to be 0.42. What has been shown is that the Launder *et al.* model of $\phi_{ij,2}$ combined with the theoretical $\phi_{ij,1}$ is compatible with both shear experiments. Hence, in our view, a large discrepancy between predictions and experiments is not due to an inadequacy of the model of $\phi_{ij,2}$, but rather to an inadequacy of the Rotta model of $\phi_{ij}^{(1)}$. An explicit confirmation of this view is given in §3.

3. Comparison with experiments

The purpose of this section is to demonstrate that a discrepancy between models and quasi-homogeneous turbulence is eliminated when the Rotta model of $\phi_{ij,1}$ is replaced by the theoretical expression, i.e. when the theoretical $\phi_{ij,1}$ is used with the Launder *et al.* model of $\phi_{ij,2}$. Specifically, we compare model predictions of relative anisotropy $m_{ij} \equiv [\langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}]/k$ with experimental values, and, for illustrative purposes, with two commonly used models. In essence, such a demonstration was made in §2.2 for the experiments of HGC and CHC. Here the demonstration will be made more explicit and for other experiments as well.

To simplify these comparisons, we use the method of algebraic modelling (Rodi 1976) to obtain an expression for m_{ij} as follows. The term $\partial\langle u_i u_j\rangle/\partial t$ is set equal to zero for the stationary condition of the experiments. The advection quantity is approximated by the algebraic-modelling assumption

$$U_1\,\partial\langle u_i u_j\rangle/\partial x_1 = k^{-1}\langle u_i u_j\rangle\,(P-\epsilon),$$

which was verified by Leslie to work very well for HGC. The pressure–strain term in (1) is obtained by substitution of (4) and (6) in (2). The consequence of all these substitutions in (1) is to produce a closed expression for $\langle u_i u_j\rangle$ that is easily solved to yield

$$m_{ij} = \frac{(1-C_{ij}^{(2)})\,(P_{ij}-\frac{2}{3}P\delta_{ij})-(\epsilon_{ij}-\frac{2}{3}\epsilon\delta_{ij})}{C_{ij}^{(1)}\,\epsilon+P-\epsilon}. \tag{15}$$

This is equivalent to the expression of Rodi, except that we have replaced c_1 and c_2 by $C_{ij}^{(1)}$ and $C_{ij}^{(2)}$ respectively, and, in addition, the extra term $\epsilon_{ij}-\frac{2}{3}\epsilon\delta_{ij}$ has been added because perfect local isotropy is not assumed. As pointed out by Rodi, the assumption implicit in (15) is not that the stresses themselves vary slowly, but rather that the relative stresses m_{ij} do. In that case m_{ij} depends only on $P/\epsilon \equiv \alpha$, the ratio of production to dissipation (Leslie 1980). To evaluate m_{ij} one need only substitute the theoretical values of $C_{ij}^{(2)}$ and $C_{ij}^{(1)}$ in (15), together with the experimental value of P/ϵ.

The values of $C_{ii}^{(2)}$ are given by the median of (11) and (12), and, as emphasized, use of these values is equivalent to use of the general Launder *et al.* model. The off-diagonal coefficient $C_{12}^{(2)}$ is 0.6.

As for the diagonal coefficients $C_{ii}^{(1)}$, the values given in figure 1 were calculated under the assumption of local isotropy. That is, to calculate $C_{ii}^{(1)}$ it was assumed that the longitudinal velocity spectrum is given by $F_i(k_1) \propto \epsilon^{\frac{2}{3}} k_1^{-\frac{5}{3}}$ ($i = 1,2,3$) in the inertial subrange, where k_1 is the longitudinal component of wavevector $\boldsymbol{k}$. However, as mentioned in §2.2 the experiments of CHC show that $F_i(k_1) \propto \epsilon_{ii}^{\frac{2}{3}} k_1^{-\frac{5}{3}}$, with $\epsilon_{11} > \epsilon_{33} > \epsilon_{22}$, in much of the inertial subrange ($2\times10^{-2} < \eta k_1 \lesssim 10^{-1}$), where η is the Kolmogoroff microscale. The values of ϵ_{ii}/ϵ were calculated in §2, and are given by (10). Although seemingly small, the deviations of ϵ_{ii}/ϵ from $\frac{2}{3}$ have a surprisingly large influence on the diagonal coefficients $C_{ii}^{(1)}$. A calculation of these coefficients for the values of ϵ_{ii}/ϵ in (10) is given in figure 2. The coefficients $C_{ii}^{(1)}$ for CHC are obtained from this figure by substitution of the observed ratios $\langle u_1^2\rangle/\langle u_2^2\rangle = 1.92$, $\langle u_1^2\rangle/\langle u_3^2\rangle = 1.67$, and the coefficients for HGC are obtained by substitution of $\langle u_1^2\rangle/\langle u_2^2\rangle = 2.52$, and $\langle u_1^2\rangle/\langle u_3^2\rangle = 1.67$ in figure 2. We obtain $C_{ii(\mathrm{CHC})}^{(1)} = (1.02, 1.32,$

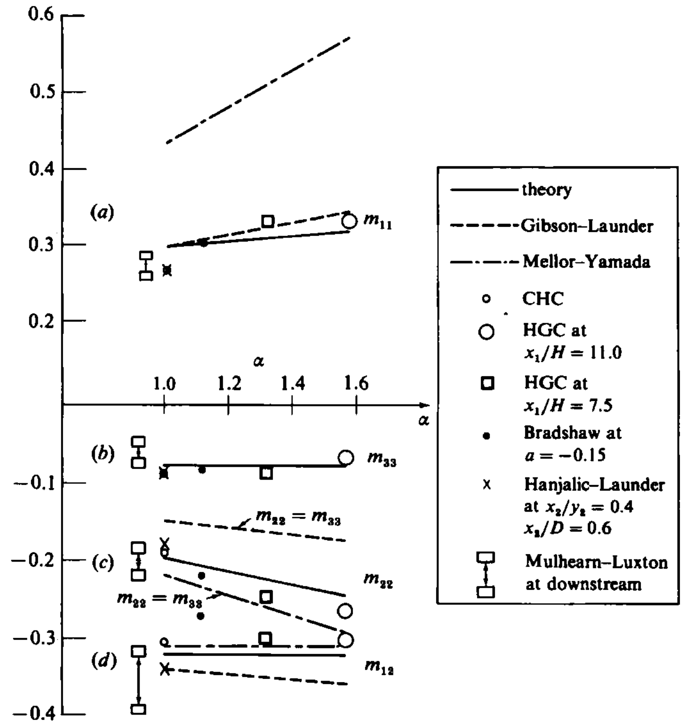

FIGURE 3. Stress components m_{ij} versus $P/\epsilon \equiv \alpha$, for the theoretical model, the abridged model (Gibson & Launder 1978) and the model used by Mellor & Yamada (1982): (*a*) m_{11}; (*b*) m_{22}; (*c*) m_{33}; (*d*) m_{12}.

0.52) for $i = 1, 2, 3$, and $C^{(1)}_{ii(\mathrm{HGC})} = (0.98, 1.17, 0.15)$. These values differ substantially from those used in models, although the average resembles that proposed by Lumley & Newman (1977) for small anisotropy. As mentioned, they are consistent with eddy-simulation values. Of particular importance is that $C^{(1)}_{33} \ll C^{(1)}_{22}$, and the former decreases by a factor of three in going from CHC to HGC conditions. The value of $C^{(1)}_{22}$ decreases by almost 15%. Only $C^{(1)}_{11}$ remains approximately the same. These values illustrate why models can correctly give m_{11} for both experiments, but not m_{22} or m_{33}, i.e. $C^{(1)}_{11}$ is fortuitously the same for both high and low shear, whereas $C^{(1)}_{22}$ and $C^{(2)}_{33}$ are not.

The predicted values of m_{ij} for both experiments are now obtained by substitution of the aforementioned $C^{(1)}_{ij}$ and $C^{(2)}_{ij}$ into (15). (Alternatively, one could use the theoretical expression for $C^{(1)}_{ii}$ instead of figure 2, and solve a more complex equation for m_{ij}. The result is the same.) The dissipation components ϵ_{ii} are given by (10) for both experiments, and we substitute $P/\epsilon = 1$ for CHC and $P/\epsilon = 1.55$ for the downstream position $x_1/h = 11.0$ of HGC. The predicted values of m_{ij} for various P/ϵ are given in figures 3(*a–d*) together with the CHC and HGC values. Several other experimental values are also included: Mulhearn & Luxton (1975), the midstream position of HGC, Hanjalić & Launder (1972) and Bradshaw (1967, with $a = -0.15$). The last two experiments, although not homogeneous as CHC, contained large regions of flow where the turbulent diffusion was very small (much smaller than P or ϵ), and consequently should satisfy (15) fairly well near a central point in the flow. For comparison, figure 3 includes the abridged model, (3) plus (5) with $c_1 = 1.8$ and $c_2 = 0.6$, the values favoured by Gibson & Launder (1978), and a particularly simplified model used for the atmospheric boundary layer (Mellor 1973; Mellor & Yamada 1982).

It is seen in figure 3 that the theoretical model agrees to within 10% with all experimental values of m_{11}, m_{22}, m_{33} and m_{12}. The abridged model agrees well with m_{11} (within 10%), less well with m_{12}, and not at all with m_{22} and m_{33} – as pointed out by Leslie (1980). The Mellor model neglects the diagonal mean-field term $\phi_{ii,2}$, without *a priori* justification, but has the virtue of simplicity. It is seen to agree with m_{22} and m_{12}, but not at all with m_{11} or m_{33}.

The extent of agreement found here between theory and experiment cannot be obtained when the Rotta constant c_1 is used for $C_{ij}^{(1)}$. One might object that three constants were needed to obtain this agreement, $C_{11}^{(2)}$, $C_{22}^{(2)}$, $C_{12}^{(2)}$, whereas the original model used only two constants, c_1 and γ. However, in actuality, only a single constant was used, namely $\gamma = 0.42$. That is, what was done in essence was to show that the experimental values of every component of $C_{ij}^{(2)}$ is in agreement (within 5%) with the Launder *et al.* model when γ is selected to be 0.42 – and this for both experiments. Hence use was made of only $\gamma = 0.42$ together with the theoretical $C_{ii}^{(1)}$.

4. Summary and conclusions

It has been demonstrated that, although inexact, the (general) Launder *et al.* model of $\phi_{ij,2}$ with $\gamma = 0.42$ is a good approximation for both strongly and weakly sheared, nearly homogeneous flows. Relatedly, this model when used together with the theoretical $\phi_{ij,1}$ predicts stress components that are all within 10% of both strong- and weak-shear experiments. The previously found discrepancy is attributed to use of the Rotta model of $\phi_{ij,1}$. Although our verification of the Launder *et al.* model is for only a simple flow condition, it is the flow most specific for the pressure–strain term. We believed that the discrepancy found between this flow and models should be explained before giving consideration to more complex flows. For such future consideration, both the Launder *et al.* model of $\phi_{ij,2}$ (given by eq. (10) of Launder *et al.*) and the theoretical $\phi_{ij,1}$ apply to more complex flows.

The theoretical $\phi_{ij,1}$ predicts that the pressure–strain term does not cause a return to isotropy. Rather, it resists large anisotropy – a weaker effect. Therefore, in addition to its consistency with the simple shear flow mentioned, it is also consistent with turbulence decay at high Reynolds number and small anisotropy. An independent verification of the theoretical $C_{ii}^{(1)}$ is provided by eddy simulations (Feiereisen *et al.* 1981) in an average sense. The formal difference between the theoretical $\phi_{ij,1}$ and the Rotta model is that the coefficients $C_{ij}^{(1)}$ are not the same for all i,j, and vary with anisotropy, but in a predictable way. The theoretical coefficients are understandably complex because they account for various lengthscales of the turbulence, and, in fact, were derived from two-point theory; i.e. the theoretical $\phi_{ij,1}$ is derived from and is equivalent to a two-point model. In that derivation, the spectral complexity inherent in a two-point description was converted by algebraic manipulation into a complex function of stress ratios contained within a single-point description. Briefly, by way of further explanation, the ratios of the many scales of turbulence spectra are accounted for by ratios of $\langle u_1^2\rangle/\langle u_2^2\rangle$ and $\langle u_1^2\rangle/\langle u_3^2\rangle$ in the single-point description given here. A different matter, but one of practical consequence, is that $C_{ij}^{(1)}$ is insensitive to the large-scale (production) region of the spectrum, and, in that sense, is universal.

Another consideration is that deviations from local isotropy needed to be accounted for in order to show that the Launder *et al.* coefficients did not vary appreciably with shear strength (i.e.for HGC were virtually the same as for CHC). Otherwise $C_{22}^{(2)}$ would have appeared to decrease by 20% as the shear strength decreased. In fact, it was $C_{22}^{(1)}$ that decreased, in a theoretically predicted way. Fortunately, CHC provided information to estimate $\epsilon_{22}/\epsilon_{33}$; otherwise the constancy of the Launder *et al.* coefficients may not have been discerned. The estimated value of $\epsilon_{22}/\epsilon_{33}$ could also have been obtained from eddy simulations (Feiereisen *et al.*).

A final note is that the full complexity of the theoretical $\phi_{ij,1}$ was included because our principal aim was to present the theory and determine if it could explain the CHC and HGC experiments. The theoretical expression might be simplified for use in models if a need should arise in the future.

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Appendix. The coefficients $C_{ii}^{(1)}$ derived in paper II

The purpose of the Appendix is to present the expressions for these coefficients as explicit functions of $\langle u_1^2\rangle/\langle u_2^2\rangle$ and $\langle u_1^2\rangle/\langle u_3^2\rangle$ and to outline how they are calculated from equation (C 3) of that paper. In II, $C_{ii}^{(1)}$ was actually given in terms of other dimensionless coefficients β_{ii} as follows:

$$C_{11}^{(1)} \equiv \frac{\beta_{11}-\frac{1}{2}Y^{-\frac{2}{3}}\beta_{33}-\frac{1}{2}Z^{-\frac{2}{3}}\beta_{22}}{a_{11}/\langle u_1^2\rangle} \equiv -\frac{k}{\epsilon}\frac{\phi_{11,1}}{a_{11}} \tag{A 1}$$

$$C_{22}^{(1)} \equiv \frac{Z^{-\frac{2}{3}}\beta_{22}-\frac{1}{2}\beta_{11}-\frac{1}{2}Y^{-\frac{2}{3}}\beta_{33}}{a_{22}/\langle u_1^2\rangle} \equiv -\frac{k}{\epsilon}\frac{\phi_{22,1}}{a_{22}} \tag{A 2}$$

$$C_{33}^{(1)} \equiv \frac{Y^{-\frac{2}{3}}\beta_{33}-\frac{1}{2}\beta_{11}-\frac{1}{2}Z^{-\frac{2}{3}}\beta_{22}}{a_{33}/\langle u_1^2\rangle} \equiv -\frac{k}{\epsilon}\frac{\phi_{33,1}}{a_{33}} \tag{A 3}$$

where $a_{ii} \equiv \langle u_i^2\rangle - \frac{2}{3}k$ is the anisotropy, $Y^{\frac{2}{3}} \equiv \langle u_1^2\rangle/\langle u_3^2\rangle$, and $Z^{\frac{2}{3}} \equiv \langle u_1^2\rangle/\langle u_2^2\rangle$. The coefficient β_{22} is given by (C 3) of II as an integral of the scalar spectrum $E_{22}(k)$. This expression also gives β_{11} and β_{33} when $E_{22}(k)$ is replaced by $E_{11}(k)$ and $E_{33}(k)$ respectively. The expression (C 3) can be evaluated by straightforward integration when $E_{ii}(k)$ are specified. For the case of large Reynolds number and local isotropy, it was justified in II to use $E_{ii}(k) = \frac{2}{3}\alpha\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ for $k_t \leqslant k \leqslant k_\nu$, $E_{ii}(k) = \frac{2}{3}\alpha\epsilon^{-\frac{2}{3}}k_t^{-\frac{5}{3}-m}$ for $k < k_t$, where $\alpha = 1.5$ is the Kolmogoroff constant, $m > -1$ is an adjustable parameter justified to be 2, k_ν is the viscous 'cut-off' wavenumber, and k_t is an energy-containing wavenumber determined by $\langle u_i^2\rangle \equiv \int \mathrm{d}k\, E_{ii}(k) = \alpha\epsilon^{\frac{2}{3}}k_t^{-\frac{2}{3}}[1+\frac{2}{3}(m+1)^{-1}]$ in terms of $\langle u_i^2\rangle$ and ϵ. It can be seen that k_t is the wavenumber where $E_{ii}(k)$ is maximum. In II the β_{ii} integrals were evaluated for the special case $\langle u_2^2\rangle = \langle u_3^2\rangle$. Here we consider the general case $\langle u_2^2\rangle \neq \langle u_3^2\rangle$. Also, (C 4)–(C 8) of II are not accurate enough for computations, and have misprints.

To recalculate β_{ii} for local isotropy, we substitute the above expression for $E_{ii}(k)$

and $E \equiv \frac{1}{2}(E_{11}+E_{22}+E_{33})$ in (C 3), and numerically integrate for various Y and Z. For $Z \geqslant Y \geqslant 1$ these integrated values of β_{ii} are accurately given by

$$\beta_{11} = DQ\left[(1+Z^{-\frac{1}{3}}+Y^{-\frac{1}{3}})\frac{0.72}{2^{\frac{3}{2}}}+(1-Y^{-\frac{2}{3}})(1-Y^{-\frac{1}{3}})H(1,Y) + (1-Z^{-\frac{2}{3}})(1-Z^{-\frac{1}{3}})H(1,Z)\right], \quad \text{(A 4)}$$

$$\beta_{22} = DQ\left[Z^{\frac{1}{3}}\frac{2.16}{2^{\frac{3}{2}}}+(1-Z^{-\frac{2}{3}})(1-Z^{-\frac{1}{3}})H(Z,1)+(Z^{-\frac{2}{3}}-Y^{-\frac{2}{3}})\left(\frac{Y^{\frac{1}{3}}}{Z^{\frac{1}{3}}}-1\right)H(Z,Y)\right], \quad \text{(A 5)}$$

$$\beta_{33} = DQ\left[\left(2+\frac{Y^{\frac{1}{3}}}{Z^{\frac{1}{3}}}\right)Y^{\frac{1}{3}}\frac{0.72}{2^{\frac{3}{2}}}+(1-Y^{-\frac{2}{3}})(1-Y^{-\frac{1}{3}})H(Y,1) + (Y^{-\frac{2}{3}}-Z^{-\frac{2}{3}})\left(1-\frac{Y^{\frac{1}{3}}}{Z^{\frac{1}{3}}}\right)H(Y,Z)\right], \quad \text{(A 6)}$$

$$H(a,b) \equiv \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\left[2-\frac{2.4b^2}{a^2+b^2}-0.08\left(\frac{4ab}{a^2+b^2}-1\right)+\frac{2}{3}\left(1-\frac{2a^2}{a^2+b^2}\right)\right],$$

$$Q \equiv (1+Y^{-\frac{2}{3}}+Z^{-\frac{2}{3}})^{\frac{1}{2}},$$

$$D \equiv 1.81(1+R_\nu^{-\frac{3}{4}}-R_\nu^{-\frac{1}{4}})(1-R_\nu^{-\frac{1}{2}}),$$

$$R_\nu \equiv \frac{(\frac{2}{3}k)^{\frac{1}{2}}}{\nu k_0}, \quad k_0 = 4.5\frac{\epsilon}{k^{\frac{2}{3}}},$$

where R_ν is the Reynolds number. These expressions determine $C_{ii}^{(1)}$ explicitly in terms of stress ratios $Y^{\frac{2}{3}} = \langle u_1^2\rangle/\langle u_3^2\rangle$ and $Z^{\frac{2}{3}} = \langle u_1^2\rangle/\langle u_2^2\rangle$ – for the case of local isotropy. Values of $C_{ii}^{(1)}$ are plotted in figure 1.

When deviations from local isotropy are significant, the coefficients β_{ii} are calculated by simply replacing $E_{ii} \propto \epsilon^{\frac{2}{3}}$ with $E_{ii} \propto \epsilon_{ii}^{\frac{2}{3}}$, and integrating (C3). The resulting values of $C_{ii}^{(1)}$ for the CHC and HGC experiments, where, approximately, $\epsilon_{11} = 1.25\epsilon_{22}$ and $\epsilon_{33} = 1.13\epsilon_{22}$, are plotted in figure 2 for $\langle u_1^2\rangle/\langle u_3^2\rangle = 1.67$, the stress ratio found in both experiments.

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